Optimal production lots for items with imperfect production and screening processes without using derivatives

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Abstract: Complicated traditional computing processes take the first and second order derivatives to obtain the EOQ/EPQ formula that are difficult to understand. This study proposes an EPQ model for imperfect items, which utilises a simple computing method without using derivatives and considers different screening rates to obtain the optimal production lots. The analytical method for sensitivity analysis is examined. A numerical example is provided to illustrate the proposed model. Using the proposed model, researchers who are unfamiliar with calculus may be able to find a solution to procedures with inventory problems more easily.

Keywords: imperfect quality; screening process; derivatives.


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1 Introduction

The role of optimal production lots in the inventory and production models is widely discussed in literature and among practitioners. Salameh and Jaber (2000) considered an inventory model for defective items using the EPQ/EOQ formula. Later, Khan et al. (2011) developed an economic order quantity for items with imperfect quality and inspection errors. Chiu et al. (2012) combined an alternative multi-delivery policy into an imperfect economic production quantity model with partial rework. Lee and Kim (2014) developed an integrated production-distribution model to determine an optimal policy with both deteriorating and defective items under a single-vendor single-buyer system. Researchers as well as practitioners focusing on issues related to defective items have repeatedly stressed the importance of developing solutions for inventory control problems.

The application of calculus by taking the first and second order derivatives of the objective function is the most popular approach to derive the EOQ/EPQ formula. However, due to the complicated traditional computing process being difficult to understand, the without derivatives method has popularised in recent years (Ronald et al., 2004; Jason et al., 2005; Minner, 2007; Wee et al., 2009; Teng, 2009). There are three methodologies of Without Derivatives in existing literature:

1. complete square method (Grubbström and Erdem, 1999; Leung, 2008)
2. cost difference comparisons method (Minner, 2007; Wee et al., 2009)
3. arithmetic-geometric-mean-inequality method (Teng, 2009).

Chen et al. (2012) derived an optimal replenishment lot size, and a simplified optimal production-inventory cost formula for such a particular EPQ model can be derived without derivatives. Based on the methods mentioned above, most researchers who are unfamiliar with calculus may be able to understand the solution to procedures with inventory problems more easily.

Sensitivity analysis is essential in deriving insights from decision-support models in a wide range of applications (Borgonovo, 2010). Koltai and Terlaky (2000) showed that managerial questions received no satisfactory answers from the mathematical interpretation of sensitivity analysis when the solution of a linear programming model is degenerated. Singh et al. (2005) studied multi-parametric sensitivity analysis for programming problems with linear-plus-linear fractional objective function using the concept of maximum volume in the tolerance region. Borgonovo (2010) defined sensitivity measures that did not rest on differentiability and related the sensitivity measures to classical differential with comparative statics indicators, then proved
a result that allowed us to obtain the sensitivity measures at the same cost of one-variable-at-a-time methods, thus making their estimation feasible also for computationally intensive models. Khanra et al. (2014) discussed sensitivity analysis in a newsvendor model. In previous studies, sensitivity analysis was always based on discrete values of parameter (Weng, 2003; Hsu et al., 2009), while continuous values received little attention. This study develops an inventory model for imperfect production processes based on the study of Salameh and Jaber (2000) to obtain the economic production quantity without using derivatives. The analytical method for sensitivity analysis is examined. Numerical example is provided to illustrate the proposed model.

2 Assumptions and notation

For convenience, the assumptions and notations from Salameh and Jaber (2000) are adopted.

The mathematical model presented in this study has the following assumptions:

1. The demand rate is known and constant.
2. The production rate is known and constant.
3. The lead-time is known and constant.
4. The screening process and demand proceeds simultaneously, and the screening rate, \( x \), is greater than the demand rate, \( D \), i.e., \( x > D \).
5. A single product is considered.

Meanwhile, the mathematical models have the following notations:

- \( T \) the production cycle length
- \( t_s \) the screening time per cycle
- \( t_1 \) the production run time per cycle, decision variable
- \( D \) the demand rate per year
- \( M \) the production rate, \( M > D \)
- \( x \) the screening rate, \( x > D \)
- \( c \) the production cost per unit
- \( K \) the setup cost per production
- \( P \) the defective percentage per production, random variable, which is uniformly distributed over \([\alpha, \beta]\), where \( 0 \leq \alpha \leq \beta < 1 \).
- \( f(p) \) the probability density function of \( p \)
- \( s \) the selling price of good quality items per unit
- \( v \) the selling price of defective items per unit, \( v < c \)
- \( D \) the screening cost per unit
- \( H \) inventory holding cost per item
Optimal production lots for items with imperfect production

* the superscript representing optimal value

TR the total revenue per cycle; which is the sum of total sales of good quality and imperfect quality items

TC the total cost per cycle

TPU the net profit per unit time

ETPU the expected value of TPU.

3 Analysis of the model

This study assumes an imperfect production process with the constant production rate of $M$, the production cost of $c$ per unit, and a setup cost of $K$ per production. Each lot produced contains a certain percentage of defectives, with known probability density function $f(p)$, where $p$ is the defective percentage. The selling price of good quality item is $s$ per unit. For items with imperfect quality, it is assumed that 100% screening process of the production is conducted at a constant rate of $x$ per unit time; and items with poor quality are kept in stock and sold prior to the next production as a single batch at a discount price of $v$. Two cases may occur here:

1. When the screening rate is higher than the production rate (i.e., $x \geq M$), the screening time of $t_s$ is exactly the production run time of $t_1$; the behaviour of the inventory level is illustrated in Figure 1.

2. When the screening rate is lower than the production rate (i.e., $x < M$), the screening time of $t_s$ is longer than the production run time of $t_1$; the behaviour of the inventory level is illustrated in Figure 2.

The optimal operating inventory strategy is derived from the trade off among the total revenue per unit time, the production cost per unit time, the inventory holding cost per unit time, and item screening cost per unit time. Therefore, the sum will be maximum.

Figure 1 Inventory system when $x \geq M$
3.1 Case I. When $x \geq M$

To avoid shortages occurring at $t_1 (= t_s)$ for the feasibility of model, it is assumed that the number of goods item produced, $(1 - p)M_{t_1}$, is at least equal to the demand, $D_{t_1}$, during production time $t_1$, that is $(1 - p)M_{t_1} - D_{t_1} \geq 0$, or

$$p \leq 1 - \frac{D}{M}. \quad (1)$$

The random variable $p$ is uniformly distributed over $[\alpha, \beta]$, where $0 \leq \alpha \leq \beta < 1$. Define $TR(t_1, p)$ as the total revenue, which is the sum of total sales of both goods and imperfect quality items. One has

$$TR(t_1, p) = (1 - p)M_{t_1}s + pM_{t_1}v. \quad (2)$$

$TC(t_1, p)$ is the sum of setup cost per unit time, production cost per unit time, screening cost per unit time, and holding cost per unit time. One has the following formula

$$TC(t_1, p) = K + M_{t_1}c + M_{t_1}d + h \left[ \frac{1}{2} (M - D)t_1^2 + \frac{(T - t_1)(M - D)t_1 - pM_{t_1}}{2} \right]$$

$$= K + M_{t_1}c + M_{t_1}d + h \left[ \frac{1}{2} (M - D)t_1^2 + \frac{(M - D)t_1 - pM_{t_1}}{2D} \right]^2 \quad (3)$$

where

$$T = \frac{(1 - p)M_{t_1}}{D}. \quad (4)$$

The total profit per unit time of $TPU(t_1, p)$ is given by dividing the total profit per cycle by the cycle length of $T$. One has

$$TPU(t_1, p) = \frac{TR(t_1, p) - TC(t_1, p)}{T}. \quad (5)$$

The expected value of $TPU(t_1, p)$ is

$$ETPU(t_1) = E \left[ \frac{TR(t_1, p) - TC(t_1, p)}{T} \right]. \quad (6)$$

Since the process generating the profit is renewal (with renewal points at production epochs), therefore, the expected profit per unit time is given by the renewal-reward theorem (Maddah and Jaber, 2008) as
The objective function of this study can be formulated as:

\[ \text{Max: } ETPU(t_1). \] (8)

Since the first term of equation (7) is constant, our problem can be transferred to minimise the following formula,

\[ \frac{DK}{(1-E(p))M} \frac{1}{M t_1} + \frac{M-D + (M-D)^2 - 2(M-D)ME(p) + M^2E(p^2)}{2D} Dht_1. \] (9)

Because \( E(p^2) = \frac{(\beta-\alpha)^2}{3} \) and \( [E(p^2)] = \left[ \frac{(\beta-\alpha)}{2} \right]^2 = \frac{(\beta-\alpha)^2}{4} \), therefore

\[
(M-D)^2 - 2(M-D)ME(p) + M^2E(p^2) = [(M-D)^2 - 2(M-D)ME(p) + M^2E(p^2)] + M^2E(p^2) - M^2E(p^2) > 0.
\]

With \( (1-E(p)) > 0 \) and \( (M-D) > 0 \), both of the two terms of formula (9) are positive. As the arithmetic-geometric mean inequality for two real positive numbers \( a \) and \( b \), one has

\[ \frac{a+b}{2} \geq \sqrt{ab}, \] (11)

with equality if and only if \( a = b \).

By using the arithmetic-geometric mean inequality, we can easily obtain
When the equality

\[
\frac{DK}{(1-E(p))M t_t} \left[ \frac{M-D + (M-D)^2 - 2(M-D)ME(p) + M^2E(p^2)}{2D} \right] \geq 2 \frac{DK}{(1-E(p))M t_t} \left[ \frac{M-D + (M-D)^2 - 2(M-D)ME(p) + M^2E(p^2)}{2D} \right] \left[ \frac{M}{1-E(p)} \right] = 2DKh \left[ (M-D)(1-2E(p)) + ME(p^2) \right] \left[ \frac{M}{1-E(p)} \right]
\]

holds, formula (9) (\(ETPU(t_1)\) as well) has an optimum. From equation (13), the optimal production run time, \(t'_1\), is

\[
t'_1 = \sqrt{\frac{2DKh[(M-D)(1-2E(p)) + ME(p^2)]}{hM[M-D] - 2(M-D)E(p) + ME(p^2)}}
\]

The economic production quantity is

\[
Q^* = M't' = \sqrt{\frac{2DKM}{h[(M-D) - 2(M-D)E(p) + ME(p^2)]}}
\]

Note that when \(p = 0\), one has

\[
Q^* = \sqrt{\frac{2DKM}{h(M-D)}} = Q^*_{\text{trad}}
\]

Hence, the optimal expected profit per unit time is

\[
ETPU(t'_1) = \frac{(1-E(p))Ds + E(p)Dv - Dc - Dd}{1-E(p)} - \sqrt{\frac{2DKh[(M-D)(1-2E(p)) + ME(p^2)]}{M}} \frac{M}{1-E(p)}
\]

### 3.2 Case II. When \(x < M\)

To avoid shortages occurring at \(t_t\) for the feasibility of model, it is assumed that the number of good item, \((1-p)Mt_1\), at least equals to the demand during \([0, t_t]\), that is, \((1-p)Mt_1 - Dt_t \geq 0\), or
Optimal production lots for items with imperfect production

\[ p \leq 1 - \frac{D}{x}, \quad (18) \]

where

\[ t = \frac{M_{tt}}{x}. \quad (19) \]

Then

\[ TR(t_1, p) = (1 - p)M_{tt}s + pM_{tt}v. \quad (20) \]

\[ TC(t_1, p) = K + cM_{tt} + dM_{tt} + h \left\{ (M - D)t_1 \frac{f_1}{2} \right. \]
\[ + \left[ 2(M - D)t_1 - D(t_2 - t_1) \right] \frac{(t_2 - t_1)}{2} \]
\[ + \left[ (M - D)t_1 - D(t_2 - t_1) - pM_{tt} \right] \frac{(T - t_2)}{2} \}, \quad (21) \]

where

\[ T = \frac{(1 - p)M_{tt}}{D}. \quad (22) \]

**Figure 2** Inventory system when \( x < M \)

![Inventory system](image)

The fourth term of equation (21) can be simplified as
\[ h \left\{ \frac{(M-D)t_1}{2} + \left[ \frac{2(M-D)t_1 - D(t_x - t_1)}{2} \right] \right\} \]
\[ = h \left[ M \left( Mx - Dx - 2ME(p)x + 2ME(p)D + ME(p^2)x \right) \right] t_x^2 \]
\[ = h \cdot \Delta \cdot t_x^2, \]

where
\[ \Delta = M \left[ Mx - Dx - 2ME(p)x + 2ME(p)D + ME(p^2)x \right] \left(2Dx\right). \] (23)

Then
\[ ETPU(t_1) = \frac{E\left[ TR(t_1, p) - TC(t_1, p) \right]}{E[T]} \]
\[ = \frac{(1-E(p))Ds + E(p)Dv - Dc - Dd}{(1-E(p))} \]
\[ - \frac{DK}{(1-E(p))M t_1} - \frac{\Delta}{(1-E(p))M} Dht_1. \] (24)

[Note that equation (24) is reduced to equation (7) when \( x = M \)].

The problem can be formulated as:
\[ \text{Max : } ETPU(t_1). \] (25)

Since the first term of equation (23) is constant, our problem can be transferred to minimise the following formula,
\[ \frac{DK}{(1-E(p))M t_1} + \frac{\Delta}{(1-E(p))M} Dht_1. \] (26)

For \( \Delta > 0 \) (note: \( \Delta \cdot t_x^2 \) is the area of inventory level of Figure 1 during \([0, T]\)), both terms of formula (26) are positive. By using the arithmetic-geometric mean inequality, we can easily obtain
\[ \frac{DK}{(1-E(p))M t_1} + \frac{\Delta}{(1-E(p))M} Dht_1 \geq 2 \sqrt{\frac{DK}{(1-E(p))M t_1} \frac{\Delta}{(1-E(p))M} Dht_1} \]
\[ = 2D\sqrt{K\Delta h} \]
\[ \frac{1}{(1-E(p))M}. \] (27)

When the equality
\[ \frac{DK}{(1-E(p))M t_1} = \frac{\Delta}{(1-E(p))M} Dht_1, \] (28)
holds, formula (26) has a minimum. From equation (28), the optimal production run time, \( t^* \), is

\[
t^* = \frac{K}{hDA}.
\]  

(29)

[Note that equation (29) is reduced to equation (14) when \( x = M \).]

The optimal expected profit per unit time is

\[
ETPU(t^*) = \frac{(1 - E(p))D_s + E(p)Dv - Dc - Dd}{(1 - E(p))} - \frac{2D\sqrt{KDA}}{1 - E(p)M}.
\]  

(30)

4 Numerical results and sensitivity analysis

Example 1 (Case I, when \( x \geq M \)): To validate the theory, the numerical parameters adopted from Salameh and Jaber (2000) are used:

- \( D = 50,000 \) units/year
- \( M = 120,000 \) units/year
- \( x = 1 \) unit/min
- \( K = $100/\)cycle
- \( h = $5/\)unit/year
- \( d = $0.5/\)unit
- \( c = $25/\)unit
- \( s = $50/\)unit
- and \( = $20/\)unit.

Assume that the inventory operation runs for 8 hour per day, for 360 days per year, and at the annual screening rate, \( x = 1*60*8*360 = 172,800 \) units/year. The percentage defective random variable, \( p \), can take any value in the range \([\alpha, \beta]\) with \( \alpha = 0, \) and \( \beta = 0.02 \). This study assumes that \( p \) is uniformly distributed with p.d.f. as

\[
f(p) = \begin{cases} 
50, & 0 \leq p \leq 0.02, \\
0, & \text{otherwise.}
\end{cases}
\]  

(31)

For avoiding shortages, the condition in equation (1) must be satisfied, that is,

\[
p \leq 1 - \frac{D}{M} \Rightarrow p \leq 1 - \frac{50,000}{120,000} = 0.583.
\]

From assumption, it has \( 0 \leq p \leq 0.02 \) with

\[
E(p) = \int_{\alpha}^{\beta} p f(p) dp = \int_{0}^{0.02} 50 p dp = 0.01,
\]
and

\[ E(p^2) = \int_{0}^{\beta} p^2 f(p) dp = \int_{0}^{0.02} 50p^2 dp = 0.00013. \]

Solving equation (14), the solution is \( t^*_1 = 0.016 \) year. The EPQ is \( Q^* = Mt^*_1 = 1,870 \) units and the maximum profit per year \( ETPU(t^*_1) = $1,216,806. \)

**Example 2 (Case II, when \( x \leq M \)):**

- \( D = 2,000 \) units/year
- \( M = 5,000 \) units/year
- \( x = 1 \) unit/hr
- \( K = $1,000/cycle \)
- \( h = $50/unit/year \)
- \( d = $300/unit \)
- \( c = $2,500/unit \)
- \( s = $5,000/unit \)
- \( and = $2,000/unit. \)

Assume that the inventory operation runs for 8 hour per day, for 360 days per year, and at the annual screening rate, \( x = 1*8*360 = 2,880 \) units/year. The percentage defective random variable, \( p \), is the same as Example 1. Solving equation (29), the solution is \( t^*_1 = 0.073 \) year. The EPQ is \( Q^* = Mt^*_1 = 367 \) units and the maximum profit per year \( ETPU(t^*_1) = $4,372,830. \)

### 4.1 Sensitivity analysis of \( x \)

Sensitivity analysis of \( x \) with the same parameters (except \( x \)) above is carried out by the analytical method here.

Define \( ETPUX(x) \) as the analytic function of \( x \) with unknown parameter \( x \) and the other known parameters based on \( ETPU(t^*_1) \), that is \( ETPUX(x) = ETPU(t^*_1) \) with a variable of \( x \). Then (simplified by software Maple 8)

\[
ETPUX(x) = -11.6636418\left(44,647,690x^2 + 158,475 \times 10^6 \right) \\
-221,419,313\sqrt{2,113x^2 + 375 \times 10^4} x \\
-392,959,026,800\sqrt{2,113x^2 + 375 \times 10^4} x + 140,625 \times 10^9 \right) / \\
\left[ \sqrt{2,113x^2 + 375 \times 10^4} \left(2,113x^2 + 375 \times 10^4 \right) \right]
\]

Figures 3 to 5 (using Maple 8) show the properties of \( ETPUX(x) \). Figure 3 shows the graph of \( ETPUX \), which means that \( ETPU \) increases when \( x \) increases. Figure 4 shows the graph of \( d/dx(ETPUX) \), which means that the positive slope of the curve \( ETPU \) decreases (i.e., the graph of \( ETPU \) increases slowly.) when \( x \) increases. Figure 5 shows the graph of
\( \frac{d^2}{dx^2}(ETPU) \). The increasing curve with negative value means that the concavity of \( ETPU \) diminishes when \( x \) increases.

**Figure 3**  Graph of \( ETPU \) (see online version for colours)

**Figure 4**  Graph of \( \frac{d}{dx}(ETPU) \) (see online version for colours)

**Figure 5**  Graph of \( \frac{d^2}{dx^2}(ETPU) \) (see online version for colours)
5 Conclusions

This study establishes an inventory model for imperfect production process to obtain the economic production quantity without using derivatives. The arithmetic-geometric-mean-inequality method is used to derive the optimal solution. In this study, the screening process is needed to inspect product quality. The screening rate influences the inventory holding cost. Therefore, for the constraint of model feasibility, the solution procedure of two steps is considered. Sensitivity analysis with continuous parameter values is developed for elaborative comprehension. The first and second order derivatives of objective function are presented for better understanding in the properties of those functions.

In conclusion, this approach serves as an alternative and well-designed method for the researchers.

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Optimal production lots for items with imperfect production


