Optimum Synthesis of Planar Mechanisms for Path Generation Based on a Combined Discrete Fourier Descriptor

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A two-phase synthesis method is described, which is capable of solving quite challenging path generation problems. A combined discrete Fourier descriptor (FD) is proposed for shape optimization, and a geometric-based approach is used for the scale–rotation–translation synthesis. The combined discrete FD comprises three shape signatures, i.e., complex coordinates (CCs), centroid distance (CD), and triangular centroid area (TCA), which can capture greater similarity of shape. The genetic algorithm–differential evolution (GA–DE) optimization method is used to solve the optimization problem. The proposed two-phase synthesis method, based on the combined discrete FD, successfully solves the challenging path generation problems with a relatively small number of function evaluations. A more accurate path shape can be obtained using the combined FD than the one-phase synthesis method. The obtained coupler curves approximate the desired paths quite well. [DOI: 10.1115/1.4030584]

1 Introduction

The goal of the dimensional synthesis problem for path generation of planar mechanisms (i.e., the traditional synthesis problem, herein termed the path synthesis problem) is to determine the dimensions of a definite mechanism whose coupler points can be used to trace a desired path or target points. The continuous path may be represented by a sequence of discrete points. Recently, some new procedures have been developed for solving new synthesis problems of planar linkages, such as seeking the morphing chain [1–3]. The methods for solving the traditional path synthesis problems discussed in the literature may be classified as falling into two categories: direct synthesis methods without the utilization of an atlas database [4–17], or indirect synthesis methods requiring the use of a (computerized) atlas database [18–26]. In addition to the abovementioned methods, another novel method is worthy of note for synthesizing various four-bar linkages by mapping planar displacements from Cartesian space to the image space using planar quaternions [27]. The direct synthesis method may be further divided into two subcategories: the first being one-phase synthesis [4–12] and the second two-phase synthesis [13–17]. In the two-phase synthesis method, shape synthesis is handled first followed by scale–rotation–translation synthesis. In the one-phase synthesis method, the problem is considered a mechanism optimization problem with the goal of minimizing an objective function. The most common objective function is the square deviation of the positions between the desired points and the corresponding coupler points. The technique for optimum synthesis where the square deviation of the positions is used as the objective function and the input angles of the crank serve as the design variables is based on the position information of the desired points. In other words, the technique for optimum synthesis not only attempts to simultaneously determine the shape, size, location, and orientation information of the desired path but also to obtain the input angles of the crank, i.e., the time information regarding the target points or discrete points. The number of design variables increases with the number of desired points. For example, for the path synthesis of a geared 5-bar mechanism with 40 desired points and without prescribed timing there are a total of 54 design variables. However, there are only 10 design variables for the shape of the coupler curve, 4 design variables for the size, location, and orientation of the curve, and 40 design variables for the input angles of the crank corresponding to the 40 desired points.

In some advanced optimum synthesis techniques, the shape of the curve is handled first, and this process can be separated from the other requirements to reduce the design space. One then proceeds with treatment satisfying the requirements of size, orientation, and location of the curve. In other words, this is a two-phase synthesis procedure: the first phase is known as shape synthesis (or shape optimization) and the second phase is termed scale–rotation–translation synthesis. Many techniques for shape description have been utilized in the literature due to a wide range of applications in pattern recognition, computer vision, and image processing. One of the most widely used shape descriptors is the FD, which depends upon the Fourier coefficient of the Fourier transform of the shape signature of a closed curve. With this method, a shape may be represented by a shape feature vector, which consists of the normalized FD. A variety of techniques for shape description and/or the corresponding FDS have been applied to solve path synthesis problems of mechanisms. For example, Ullah and Kota [13] proposed a two-phase synthesis method for solving the path synthesis problem, using the FD of the cumulative angular deviant function, proposed by Zhan and Rokxies [28], for shape synthesis. In that method, the objective function is then evaluated from the similarity of the curve shapes based on the deviation in the amplitude and the phase angle. After shape optimization, a sequence of ordered discrete points on the shape-optimized coupler curve analogous to the desired points (herein termed analogous points) is determined. The distance between the analogous points on the shape-optimized coupler curve and the desired points is minimized by using a structural error-like objective function to determine the optimal size, orientation, and location of the solution mechanism. In the shape description technique proposed by Dibakar and Mruthyunjaya [14], a normalization transformation technique was proposed to convert the shape of the curve to a...
canonical configuration and a global property based vector (aspect ratio, area, length, location of the centroid, and principal second moment of the curve) was proposed to measure the similarity metric for the synthesis of the external boundary of the workspaces of planar manipulators. Smaili and Diab [15] proposed a two-phase synthesis method where the shape synthesis is performed on a local description technique, the so-called cyclic angular deviation vector. After shape optimization, the analogous points are determined using a geometric-based approach to complete the second phase of synthesis, including scale synthesis using the mean distance between the centroid of the desired (shape-optimized) curve and the desired (analogous) points, rotation synthesis using the mean orientation from the centroid of the desired (shape-optimized) curve to the desired (analogous) points, and translation synthesis using the centroid of a curve. They studied two cases showing the path synthesis with 25 desired points for a 4-bar mechanism. In the two-phase synthesis method proposed by Buskiewicz et al. [16], shape synthesis is based on the curvature-based FD. They developed a mathematical procedure (translation first, followed by rotation and finally scaling) to complete the second phase of synthesis after shape optimization, which uses the properties of the centroid, the direction of the major principal axis, and the perimeter of a curve to translate, rotate, and scale the mechanism to the desired configuration. They studied three cases of path synthesis of a 4-bar mechanism. It should be noted that the incorrect order of geometric transformation (i.e., execution of translation before rotation) during the second synthesis phase will cause the locations of the synthesis mechanism and the generated path to be incorrect [29]. Thus, if translation is performed before rotation, then a second translation process should be performed after rotation because a pure rotation of the mechanism about the pivot causes rotation and translation of the coupler curve, which may be seen from the position equations of the coupler point in the world coordinate system.

Recently, Buskiewicz [17] proposed a two-phase synthesis method for handling the shape synthesis problem for a geared 5-bar mechanism which uses the function of the distance of the curve from its centroid (DCC, also termed the radial distance or CD as is used in this study) to describe the curve shape in terms of its normalized Fourier coefficients. A method for the construction of a description of the curve (F-function shape signature) without referring to the harmonic analysis is also presented. Five special paths with up to 40 discrete or target points for the geared 5-bar mechanism are studied in Ref. [17] using the CD (referred to as the DCC in Ref. [17]), F-function, and curvature-based methods. The special closed paths include a self-overlapping curve (an arc of an ellipse, denoted by Arcof-Ellipse), nonsmooth curves with straight segments and vertices (triangle and asteroid), and two sophisticated shapes (a lower-case “g,” denoted by GLetter, and a curve defined by parametric equations, denoted by ParamCurve). Such special paths have seldom been studied in path synthesis problems in the literature. They claimed that the curves ParamCurve and GLetter are best synthesized using the CD method, and the others by the curvature-based method. In Ref. [17], only the shape synthesis results are obtained. In fact, the path shape redrawn according to the obtained synthesis variables in Ref. [17] for several curves should be unsuccessful [30]. Lin [30] used an one-phase synthesis method to solve the abovementioned synthesis problems for the five special paths generated by geared 5-bar mechanisms, where the error function of the square deviation of positions is used as the objective function and the GA–DE optimization method [10] is used to solve the optimization problems. All the synthesized solutions have been validated using animation in the solidworks® assembly and confirm that the obtained mechanisms are sound and usable. Findings show that the proposed method can obtain approximately matched paths at the cost of a tremendous number of evaluations of the objective function. It can be seen that the synthesis problem for the special paths generated by a geared 5-bar mechanism is a real challenge.

To solve the challenging problem more effectively and efficiently, a two-phase synthesis method is developed in this study, where a combined discrete FD is proposed for the shape optimization and the aforementioned geometric-based approach [15,29] is used for the rotation–translation synthesis. A 2D shape may be represented by a feature function, i.e., the shape signature. CCs, curvature function, cumulative angular function, and CD are commonly used shape signatures. The shape signature is usually derived from the shape boundary coordinates and the centroid in the object-centered coordinate system. Kauppinen et al. [31] argued that the CC and CD functions have good ability to describe the shape, whereas the curvature function fails. Zhang and Lu [32] showed that shape retrieval using the FD derived from CD signature is significantly better than that using FDs derived from the other three signatures (CC, cumulative angular function, and curvature function). In 2005, Zhang and Lu [33] used another shape signature, called the area function (termed the TCA in this study), which is the area of the triangle formed by the two boundary points and the centroid of the shape, to derive FD. They found the FDs derived from CD and TCA to be the most suitable for image retrieval, whereas the FDs derived from the cumulative angular function and the curvature function are not suitable. In previous works [13,15–17] that have used the two-phase synthesis method, only one shape signature [15] or the FD derived from one shape signature [13,16,17] has been used for shape synthesis. However, the similarity of shapes obtained using a direct shape signature or FDs with one or two shape signatures is not guaranteed to be sufficient, usually being skewed (distorted) or deformed. In other words, a shape feature vector with only one or two shape signatures is not sufficient to capture the exact similarity. One probable reason for this is that the reference centroid used for the shape signature is derived from the discrete shape boundary coordinates alone so the shape signature is a discrete (partial) representation of shape features. To overcome the deficiency, we propose using a combined discrete FD that comprises the CC, CD, and TCA signatures, which contains information about the position of discrete points, the distances of the radial lines, and the areas enclosed by these radial lines. The proposed combined discrete FD is a more complete shape descriptor that can capture the shape more accurately than using a single or two discrete FDs. The corresponding three shape feature vectors are used to represent a 2D shape. The mean value of the three Euclidean distances corresponding to the three feature vectors between the desired and generated curves is used as the similarity metric and the objective function for the shape optimization. The GA–DE evolutionary algorithm [10,30] is used to solve the optimization problem.

2 Shape Optimization

2.1 Shape Signatures. In this study, the three shape signatures, i.e., CC, CD, and TCA are combined to represent a 2D shape.

2.1.1 CC Signature. The CC signature is formed by treating the positions of the boundary points \((x_m, y_m) (m = 0, 1, 2, ..., N - 1)\) as the CC and is expressed as follows:

\[
 f_m = (x_m - xc) + j(y_m - yc)
\]

where the centroid \((xc, yc)\) is computed by

\[
 xc = \frac{1}{N} \sum_{m=0}^{N-1} x_m, \quad yc = \frac{1}{N} \sum_{m=0}^{N-1} y_m
\]

where \(N\) is the number of discrete (sampling) points.

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2.1.2 CD Signature. The CD signature represents the distance between the boundary points and the centroid and is expressed as follows:

\[ f_m = \sqrt{(x_m - x_c)^2 + (y_m - y_c)^2} \]  

(3)

2.1.3 TCA Signature. The TCA signature represents the area of the triangle formed by two adjacent boundary points and the centroid and is expressed as follows:

\[ f_m = \frac{1}{2} |(x_m - x_c)(y_{m+1} - y_c) - (x_{m+1} - x_c)(y_m - y_c)| \]  

(4)

with \((x_c, y_c) = (x_0, y_0)\).

2.2 Discrete Fourier Shape Descriptor. It is not convenient to directly use the shape signature in the spatial domain to describe and match the shape of a closed curve due to the problem of invariance in the similarity transformation of a shape [34,35]. It is also troublesome to directly use a single shape signature as the objective function for shape optimization because the shape signature invariance in the similarity transformation of a shape [34,35] is also termed the magnitude of the normalized Fourier coefficient, but is not used for shape recognition because it only gives “shapeless” information such as scale or position [37]. The choice of the starting point and rotation of a shape only influence the phase of the FD. The translation of a shape does not influence the FD, with the exception of the DC component [33]. Thus, the most popular technique is to utilize the magnitude of the FD and to ignore the phase information in order to achieve rotation invariance as well as make the descriptor independent of the starting point [36]. Scale invariance can be achieved by dividing the magnitude of the FD by the maximum value of the magnitude of the FD. After the scale normalization, the magnitude of the Fourier coefficient is used as the shape descriptor. Zhang and Lu [33,35] also termed the magnitude of the normalized Fourier coefficient as the FD. The amplitudes of the FDs of the CD and TCA signatures are independent of the direction of travel; however, the direction of travel can influence the amplitudes of the FD of the CC signature. Therefore, in this study, the desired curve and the coupler curve are sampled counterclockwise.

\[ |F_{1,cc}|, |F_{2,cc}|, \ldots, |F_{N-1,cc}| \]

2.3 Similarity Metric for the Combined Fourier Descriptor. The three normalized shape feature vectors corresponding to the proposed combined shape signatures of CC, CD, and TCA are expressed by \(|F_{1,cc}|, |F_{2,cc}|, \ldots, |F_{N-1,cc}|\), \(|F_1|, |F_2|, \ldots, |F_{N-1}|\), and \(|F_{1,td}|, |F_{2,td}|, \ldots, |F_{N-1,td}|\), where \(|F_i| = (|F_i|/\max (|F_i|))\) for \(n = 0, 1, 2, \ldots, N - 1\). The sum of the three Euclidean distances with weights corresponding to the three normalized feature vectors between the desired and generated (candidate) curves is used as similarity metric and the objective function for the shape optimization is calculated as follows:

\[ f_{obj} = w_1 D_{cc} + w_2 D_{cd} + w_3 D_{td} \]  

(6)

where \(w_1, w_2, \) and \(w_3\) are the weighting factors.

\[ D_{cc} = \sqrt{\sum_{i=1}^{N-1} \left( \frac{F_{i,cc}^d - F_{i,cc}^f}{F_{i,cc}^0} \right)^2} \]  

(7)

\[ D_{cd} = \sqrt{\sum_{i=1}^{N-1} \left( \frac{F_{i,cd}^d - F_{i,cd}^f}{F_{i,cd}^0} \right)^2} \]  

(8)

\[ D_{td} = \sqrt{\sum_{i=1}^{N-1} \left( \frac{F_{i,td}^d - F_{i,td}^f}{F_{i,td}^0} \right)^2} \]  

(9)

Here, \(w_1 = w_2 = w_3 = 1/3\). In Eqs. (7)–(9), the superscripts \(d\) and \(f\) denote the desired and generated curves, respectively.

3 Synthesis Formulation for Geared 5-Bar Mechanisms

Figure 1 depicts a stick diagram and all the geometric parameters of a geared 5-bar mechanism with circular gears. Angle \(\theta_2\) is the input angle of the crank, which is measured counterclockwise. The relation between angular positions \(\theta_2\) and \(\theta_5\) of links 2 and 5, respectively, can be expressed as:

\[ \theta_5 = \lambda (\theta_2 - \theta_3) + \theta_0 \]  

(10)

where \(\lambda\) is the gear ratio, and \(\theta_3\) and \(\theta_0\) are the initial angular positions of links 2 and 5, respectively. Because link 2 always can arrive at the orientation of \(+X_r\), most researchers only utilize the initial angle of link 5 as a design variable. From Eq. (1), one may see that \(\theta_3\) combined with the gear ratio \(\lambda\) has an influence on the angular position of link 5. It is worth noting that the same geared 5-bar mechanism or the 5-bar mechanism with different initial angular positions of link 2 may cause quite different kinematic effects.

![Fig. 1 Geared 5-bar mechanism in the global coordinate system](image-url)
3.1 Design Variables. There are 14 design variables in the path generation problem, \(r_1, r_2, r_3, r_4, r_5, r_{cx}, r_{cy}, \theta_{b0}, x_0, y_0, \theta_0\) with the number of revolutions of gears 2 and 5 being denoted by \(n_2\) and \(n_5\), respectively. The variable \(\theta_0\) is associated with the orientation of the coupler curve, and \(x_0\) and \(y_0\) are associated with the location of the coupler curve. To define the basic size of the coupler curve associated with the unit length of \(r_2\), let \(r_1 = r_2/r_5\). Therefore, the final size of the coupler curve depends on the variable \(r_2\). The remaining ten variables are associated with the shape of the coupler curve. Furthermore, the values of \(n_2\) and \(n_5\) should be specified in advance \[38\], because the variation of the value of \(n_2\), related to the range of the rotation of link 2, may influence search quality (solution quality) \[30\]. Design variable vector \(X\) with \(r_2 = 1\) for the shape optimization is given as follows:

\[
X = [r_1, r_3, r_4, r_5, r_{cx}, r_{cy}, \theta_{b0}, x_0, y_0]
\]

(11)

3.2 Position Equations. Using the vector loop equation, one may have \[39\]

\[
r_2 \cos \theta_2 + r_3 \cos \theta_3 - r_4 \cos \theta_4 - r_5 \cos \theta_5 - r_1 = 0
\]

(12)

\[
r_2 \sin \theta_2 + r_3 \sin \theta_3 - r_4 \sin \theta_4 - r_5 \sin \theta_5 = 0
\]

(13)

Angles \(\theta_2\) and \(\theta_3\) can be solved from Eqs. (12) and (13). The derivation of the expressions of \(\theta_2\) and \(\theta_3\) in terms of \(\theta_2\) and \(\theta_3\) is straightforward but tedious \[39\]. Angle \(\theta_0\) is expressed as follows:

\[
\theta_3 = 2 \tan^{-1}\left(\frac{M \pm \sqrt{M^2 - 4LN}}{2L}\right)
\]

(14)

where

\[
L = K - G, \quad M = 2H, \quad N = K + G
\]

\[
G = 2r_3(r_2 \cos \theta_2 - r_5 \cos \theta_5 - r_1)
\]

\[
H = 2r_3(r_2 \sin \theta_2 - r_5 \sin \theta_5)
\]

\[
K = r_1^2 + r_2^2 + r_3^2 - r_4^2 + r_5^2 - 2r_1 r_2 \cos \theta_2
\]

\[
- 2r_5(r_2 \cos \theta_2 - r_5) \cos \theta_5 - 2r_2 r_5 \sin \theta_2 \sin \theta_5
\]

(15)

Only one configuration solution represented by the plus sign on the radical is considered in this study.

One can obtain the coordinates of the coupler point \(C\) in the coordinate system \(O_XY_z\), which are used to compute the discrete FD of the shape of the coupler curve, from Fig. 1 as follows:

\[
C_X = r_2 \cos \theta_2 + r_{cx} \cos \theta_3 - r_{cy} \sin \theta_3
\]

\[
C_Y = r_2 \sin \theta_2 + r_{cx} \sin \theta_3 + r_{cy} \cos \theta_3
\]

(16)

The coordinates of the coupler point in the world coordinate system \(OXY\) can be expressed by

\[
\begin{bmatrix}
C_X \\
C_Y
\end{bmatrix} = \begin{bmatrix}
\cos \theta_0 & -\sin \theta_0 \\
\sin \theta_0 & \cos \theta_0
\end{bmatrix} \begin{bmatrix}
C_X \\
C_Y
\end{bmatrix} + \begin{bmatrix}
x_0 \\
y_0
\end{bmatrix}
\]

(17)

3.3 Uniform Sampling Points for a Closed Curve. The arc length of the coupler curve from a starting point with the input angle \(\theta_2\) to a point with the input angle \(\theta_2\) is expressed as follows:

\[
s(\theta_2) = \int_{\theta_0}^{\theta_2} \sqrt{(C_X')^2 + (C_Y')^2} d\theta_2
\]

\[
= \int_{\theta_0}^{\theta_2} \sqrt{(r_{cx}^2 + r_{cy}^2)(\theta_3')^2 + 2r_2[r_{cx} \cos(\theta_2 - \theta_3) + r_{cy} \sin(\theta_2 - \theta_3)]\theta_3'} d\theta_2
\]

\[
\theta_3' = \frac{d\theta_3}{d\theta_2}
\]

\[
= \frac{2r_2 r_3 \sin(\theta_3 - \theta_2) + 2r_3 r_5 \sin(\theta_3 - \theta_5) + 2(\lambda - 1)r_2 r_5 \sin(\theta_5 - \theta_2) + 2r_1 r_2 r_5 \sin(\theta_5 - \theta_2) - \lambda r_5 \sin \theta_3}{2r_2 r_3 \sin(\theta_3 - \theta_2) + 2r_3 r_5 \sin(\theta_3 - \theta_5) - 2r_1 r_2 r_5 \sin \theta_3}
\]

(18)

(19)

Let the input angle corresponding to the starting point \(P_0\) (i.e., the first sampling point of the coupler curve) be just at the initial angular position of link 2 (i.e., \(\theta_2 = \theta_{b0}\)). The other points are sampled in an equidistant manner, and the arc length between two adjacent sampling points is \(S/N\), where \(S = s(\theta_{b0} + 2n_{b2})\) is the perimeter of the coupler curve. Now, the values of \(\theta_2\) corresponding to the other sampling points \(P_i\) \((i = 1, 2, ..., N - 1)\) can be obtained using the bisection method for the arc length, and then the coordinates of the sampling points can be obtained using Eqs. (10) and (14)–(16). To verify whether the traveling direction is counterclockwise, the inequality is checked as follows:

\[
P_0 P_a \times P_0 P_b = \mathbf{k} \quad \text{with} \quad z > 0
\]

(20)

where \(\mathbf{k}\) is the unit vector of the \(Z\) axis (not shown in Fig. 1); \(P_a\) and \(P_b\) are the position vectors of the \(a\)th and \(b\)th points, respectively. The choice of the \(a\)th and \(b\)th points depends on the desired curve and the discrete points. In this work, \(a = N/4\) and \(b = N/2\). If \(z < 0\), the traveling direction is clockwise, and the original sampling points \(P_i\) \((i = 1, 2, ..., N - 1)\) should be replaced by \(P_{N-i}\). If necessary, the farthest point can be used as the starting point for the desired and coupler curves to assure that Eq. (20) can correctly determine the direction.

3.4 Analogous Points for Unequally Spaced Target Points. The shape signature for the FD does not necessarily sample the boundary in an equidistant manner \[40\]. However, sampling points on the coupler curve and target points should be corresponding. In other words, the sampling points should be the so-called analogous points, originally used in the second synthesis phase. Therefore, with the help of the determination of analogous points, the proposed shape synthesis method using the combined FD can be extended to the synthesis problems with unequally spaced target points. The determination of analogous points is described below. The farthest point from the centroid is chosen as the starting point for the target points and the coupler curve.

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If there are two farthest points, the choice of the starting point should satisfy the counterclockwise traveling direction using the two vectors formed from the centroid to the farthest points. If there are more than two farthest points for double symmetrical curves or regular polygonal curves, any farthest point can be determined according to the aforementioned method. The geometric-based approach proposed by Smaili and Diab [15,29] is utilized to determine the orientation [37]. The farthest point from the centroid is used as the first analogous (discrete) point for the generated (desired) curve and the other analogous points are determined according to the aforementioned method. The geometric-based synthesis using the mean orientation from the centroid of the desired (shape-optimized) curve and the desired (analogous) points, rotation synthesis using the mean orientation from the centroid of the desired (shape-optimized) curve to the desired (analogous) points, and translation synthesis using the centroid of a curve.

If a similar curve is a mirror reflection of the desired curve, the scale–rotation–translation synthesis approach will fail to achieve the final match between the similar curve and the desired curve. To achieve the final match, the desired curve first has to be mirrored with respect to the Y-axis, after which the analogous points are found and scale–rotation–translation synthesis is performed. Finally, the synthesis mechanism has to be mirrored with respect to the Y-axis, as shown in Fig. 2. The mirror-image mechanism is the final synthesis mechanism.

5 Implementation of the Proposed Two-Phase Synthesis Method

The procedure for the proposed two-phase synthesis method is described below:

1. Compute the discrete FD for the target (or discrete) points.
2. Generate initial guesses for shape optimization.
3. Find the analogous (or uniformly sampling) points.
4. Compute the discrete FD for the analogous (or uniformly sampling) points.
5. Compute the value of the objective function.

![Fig. 2 Final synthesis mechanism when the coupler curve (not shown) is similar to the mirror-image one of the desired curve](image-url)
is used to solve the optimization problem. The user-supplied parameters for the GA–DE evolutionary algorithm are described below. A population size of \( N_p = 100 \) is used, and the number of generations is \( G_{\text{max}} = 1000 \) for all problems. The top two individuals for ParaCurve and Gletter curves are used as the disturbed vectors. The major perturbation rate is \( P_{\text{maj}} = 0.6 \) and the minor perturbation rate is \( P_{\text{min}} = 0.05 \). Twenty repeated runs are executed for all problems to find satisfactory results. The angle magnitude for all results shown in the tables below is expressed in radians.

### 6 Results

Five path synthesis problems for the special paths, including triangle, asteroid, ParamCurve, ArcEllipse, and GLetter curves, are studied. For all problems discussed here, let \( n_2 = 1, 2, 3 \) and \( n_3 = \pm 1 \). Thus, there are six cases that need to be performed to find the best values of \( n_2 \) and \( n_3 \) for one problem. The limits of the design variables are \( r_1, r_3, r_5, r_3 \in [0, 6, 20]; r_6, r_7 \in [-20, 20]; \) and \( \theta_{20}, \theta_0 \in [0, 2\pi] \). The GA–DE evolutionary algorithm [10,30] is used to solve the optimization problem. The user-supplied parameters for the GA–DE evolutionary algorithm are described below. A population size of \( N_p = 100 \) is used, and the number of generations is \( G_{\text{max}} = 1000 \) for all problems. The top four individuals for triangle, asteroid, and ArcEllipse curves and the top two individuals for ParaCurve and GLetter curves are used as the disturbed vectors. The major perturbation rate is \( P_{\text{maj}} = 0.6 \) and the minor perturbation rate is \( P_{\text{min}} = 0.05 \). Twenty repeated runs are executed for all problems to find satisfactory results. The angle magnitude for all results shown in the tables below is expressed in radians.

### Table 1 Discrete points for problem 1 (triangle)

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<td>3</td>
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### Table 2 Shape synthesis results for problem 1 (triangle)

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<td>( N^m )</td>
<td>100,000</td>
<td>800,000</td>
<td>1,129,480</td>
<td>793,155</td>
<td>707,740</td>
</tr>
<tr>
<td>( r_1 )</td>
<td>15.94176</td>
<td>12.46393</td>
<td>12.03</td>
<td>12.996</td>
<td>8.66072</td>
</tr>
<tr>
<td>( r_2 )</td>
<td>4.944023</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( r_3 )</td>
<td>16.80694</td>
<td>19.52292</td>
<td>11.06</td>
<td>17.917</td>
<td>16.1321</td>
</tr>
<tr>
<td>( r_4 )</td>
<td>7.363424</td>
<td>16.98082</td>
<td>14.05</td>
<td>10.9</td>
<td>16.0319</td>
</tr>
<tr>
<td>( r_5 )</td>
<td>5.480886</td>
<td>4.977621</td>
<td>4.897</td>
<td>3.909</td>
<td>3.29279</td>
</tr>
<tr>
<td>( r_6 )</td>
<td>2.965595</td>
<td>17.29110</td>
<td>2.1231</td>
<td>-7.8541</td>
<td>-8.6612</td>
</tr>
<tr>
<td>( r_7 )</td>
<td>-3.601362</td>
<td>5.345553</td>
<td>2.8912</td>
<td>-5.3320</td>
<td>0.5454</td>
</tr>
<tr>
<td>( \theta_{20} )</td>
<td>0.0118</td>
<td>0.0212</td>
<td>0.0180</td>
<td>0.0294</td>
<td>0.2672</td>
</tr>
<tr>
<td>( \theta_0 )</td>
<td>0.0549</td>
<td>0.0888</td>
<td>0.0779</td>
<td>0.1220</td>
<td>0.3987</td>
</tr>
<tr>
<td>( \mu )</td>
<td>0.0283</td>
<td>0.0470</td>
<td>0.0426</td>
<td>0.0629</td>
<td>0.3372</td>
</tr>
</tbody>
</table>

### Table 3 Scale–rotation–translation synthesis results for problem 1 (triangle)

<table>
<thead>
<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>( r_3 )</td>
<td>3.706966</td>
</tr>
<tr>
<td>( \theta_0 )</td>
<td>0.368726</td>
</tr>
<tr>
<td>( x_0 )</td>
<td>-13.55740</td>
</tr>
<tr>
<td>( y_0 )</td>
<td>4.254157</td>
</tr>
</tbody>
</table>

### 6.1 Problem 1: Triangle Curve With 21 Unequally Spaced Discrete Points

The discrete points for the triangle curve are shown in Table 1. The number \( (N^m) \) of evaluations of the objective function, the synthesis design variables, the Euclidean distances of the FDs for the three shape signature, and the value of the objective function are shown in Table 2, together with the synthesis solutions obtained by Lin [30] and Buskiewicz [17]. The dimensions for Lin [30] in Table 2 are real, not normalized. The scale–rotation–translation synthesis results are shown in Table 3. Figure 3 shows the desired path and the coupler curve obtained using the proposed two-phase synthesis method. For the purpose of comparison, the desired path and the coupler curve obtained by Lin [30] using the one-phase synthesis method are shown in Fig. 4; the scale is the same. The present coupler curve approximates the desired path quite well. The shape similarity of the synthesis results obtained using the proposed two-phase synthesis method is more accurate than that obtained using the one-phase synthesis method for problem 1 in Ref. [30].

### 6.2 Problem 2: Asteroid Curve With 40 Equally Spaced Discrete Points

The Asteroid curve is defined by the parametric equation \( x(t) = 5 \cos(t)(2\pi), \ y(t) = 5 \sin(t)(2\pi), \ t \in [0,1] \). The curve is uniformly sampled at 40 discrete points for shape optimization. The number of evaluations of the objective function, the synthesis design variables, the Euclidean distances of the FDs are shown in Table 1. The desired path and the coupler curve obtained using the proposed two-phase synthesis method for problem 1 are shown in Fig. 3. The scale is the same. The present coupler curve approximates the desired path quite well. The shape similarity of the synthesis results obtained using the proposed two-phase synthesis method is more accurate than that obtained using the one-phase synthesis method for problem 1 in Ref. [30].

![Fig. 3 Desired path (triangle) and the coupler curve obtained using the proposed two-phase synthesis method for problem 1](image-url)
for the three shape signature and the value of the objective function are shown in Table 4, together with the synthesis solutions obtained by Lin [30] and Buśkiewicz [17]. The scale–rotation–translation synthesis results are shown in Table 5.

**Table 4 Shape synthesis results for problem 2 (asteroid)**

<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>$N^s$</td>
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<td>1,551,275</td>
<td>821,535</td>
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<td>$r_1$</td>
<td>0.600000</td>
<td>1.000005</td>
<td>1.130413</td>
<td>1.29</td>
</tr>
<tr>
<td>$r_2$</td>
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<td>1.330413</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$r_3$</td>
<td>19.85938</td>
<td>19.80421</td>
<td>17.762</td>
<td>19.927</td>
</tr>
<tr>
<td>$r_4$</td>
<td>2.199689</td>
<td>2.941310</td>
<td>5.33</td>
<td>13.1666</td>
</tr>
<tr>
<td>$r_5$</td>
<td>19.99077</td>
<td>20.00000</td>
<td>15.73</td>
<td>1.436</td>
</tr>
<tr>
<td>$r_{xy}$</td>
<td>0.615522</td>
<td>3.094287</td>
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<td>-4.6316</td>
</tr>
<tr>
<td>$r_{xy}$</td>
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<td>2.090231</td>
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<td>-5.9543</td>
</tr>
<tr>
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<td>0.608659</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
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<td>2.070137</td>
<td>5.361312</td>
<td>3.265</td>
<td>2.7037</td>
</tr>
<tr>
<td>$n_2$</td>
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<td>3</td>
<td>3</td>
<td>-3</td>
</tr>
<tr>
<td>$n_5$</td>
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<td>-1</td>
<td>-1</td>
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<tr>
<td>$D_{cc}$</td>
<td>0.0151</td>
<td>0.0298</td>
<td>0.0665</td>
<td>1.1233</td>
</tr>
<tr>
<td>$D_{cd}$</td>
<td>0.0114</td>
<td>0.0234</td>
<td>0.0470</td>
<td>0.1866</td>
</tr>
<tr>
<td>$D_{cd}$</td>
<td>0.0418</td>
<td>0.1017</td>
<td>0.1605</td>
<td>0.2376</td>
</tr>
<tr>
<td>$f_{obj}$</td>
<td>0.0228</td>
<td>0.0516</td>
<td>0.0913</td>
<td>0.5158</td>
</tr>
</tbody>
</table>

**Table 5 Scale–rotation–translation synthesis results for problem 2 (asteroid)**

<table>
<thead>
<tr>
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<th></th>
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</thead>
<tbody>
<tr>
<td>$n_2$</td>
<td>1.254219</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n_5$</td>
<td>-3.047502</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$y_0$</td>
<td>0.001823</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$\theta_0$</td>
<td>0.023848</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

for the three shape signature and the value of the objective function are shown in Table 4, together with the synthesis solutions obtained by Lin [30] and Buśkiewicz [17]. The scale–rotation–translation synthesis results are shown in Table 5. Figure 5 shows the desired path and the coupler curve obtained using the proposed two-phase synthesis method. For the purpose of comparisons, the desired path and the coupler curve obtained by Lin [30] using the one-phase synthesis method are shown in Fig. 6; the scale is the same. The present coupler curve approximates the desired path quite well. The shape similarity of the synthesis results obtained using the proposed two-phase synthesis method is more accurate than that obtained using the one-phase synthesis method for problem 2 in Ref. [30].
with the synthesis solutions obtained by Lin [30] and Bułakiewicz [17]. The scale–rotation–translation synthesis results are shown in Table 7.

The obtained coupler curve is similar to the one-phase synthesis method. For the purpose of comparisons, the desired path and the coupler curve obtained by Lin [30] using the one-phase synthesis method are shown in Fig. 8; the scale is the same. The present coupler curve approximates the desired path quite well. The shape similarity of the synthesis results obtained using the proposed two-phase synthesis is not obviously superior to that obtained using the one-phase synthesis method for problem 3 in Ref. [30]. We employed the DE algorithm and/or the least squares of the three Euclidean distances as the objective function to solve the problem. However, there was no improvement in the results of $D_{cc}$, $D_{cd}$, and $D_{icd}$.

### 6.4 Problem 4: ArcofEllipse Curve With 40 Equally Spaced Discrete Points

The ArcofEllipse curve is defined by the following parametric equation: $x(t) = 3 \cos((\pi/6) + 2\pi t)$, $y(t) = 2 \sin((\pi/6) + 2\pi t)$, $t \in [0,0.5]$. The arc is traced for $t \in [0,0.5]$ and then back to the starting point along the same path. There are 21 and 19 sampling points with the same spacing along the forward and backward paths, respectively. The number of evaluations of the objective function, the synthesis design variables, the Euclidean distances of the FPs for the three shape signatures, and the value of the objective function are shown in Table 8, together with the synthesis solutions obtained by Lin [30] and Bułakiewicz [17]. The obtained coupler curve is similar to the mirror-image curve of the original desired path. As mentioned previously, to achieve a final match, the desired curve first has to be mirrored with respect to the $Y$-axis, after which scale–rotation–translation synthesis is performed. The synthesis results are shown in Table 9. Figure 9 shows the mirror-image curve of the desired path and the coupler curve obtained using the proposed two-phase synthesis method. For the purpose of comparisons, the desired path and the coupler curve obtained by Lin [30] using the one-phase synthesis method are shown in Fig. 10; the scale is the same. The present coupler curve approximates the desired path quite well. The shape similarity of the synthesis results obtained using the proposed two-phase synthesis is more accurate than that obtained using the one-phase synthesis method.

### Table 7 Scale–rotation–translation synthesis results for problem 3 (ParamCurve)

<table>
<thead>
<tr>
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<th></th>
<th></th>
<th></th>
</tr>
</thead>
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<td>3.040992</td>
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<td>2.757110</td>
<td>-25.19989</td>
<td>1.000081</td>
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<td>1</td>
<td>1</td>
<td>7.248557</td>
<td>25.19989</td>
<td>9.132569</td>
<td>10.15520</td>
<td>18.727</td>
<td>17.463</td>
<td>9.853775</td>
</tr>
<tr>
<td>-1.172489</td>
<td>-10.36374</td>
<td>7.6197</td>
<td>4.7261</td>
<td>2.397015</td>
<td>4.5475</td>
<td>6.7476</td>
<td>2.4994</td>
<td>9.9053</td>
</tr>
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</table>

### Table 8 Shape synthesis results for problem 4 (ArcofEllipse)

<table>
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<th></th>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.387188</td>
<td>-0.558232</td>
<td>-0.558645</td>
<td>-1.420063</td>
<td>0.0319</td>
<td>0.0671</td>
<td>0.1746</td>
<td>0.3648</td>
<td>0.1968</td>
</tr>
<tr>
<td>0.0057</td>
<td>0.0493</td>
<td>0.0583</td>
<td>0.0648</td>
<td>0.0332</td>
<td>0.0920</td>
<td>0.1829</td>
<td>0.4216</td>
<td>0.1968</td>
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<tr>
<td>0.0057</td>
<td>0.0493</td>
<td>0.0835</td>
<td>0.0648</td>
<td>0.0312</td>
<td>0.0920</td>
<td>0.1829</td>
<td>0.4216</td>
<td>0.1968</td>
</tr>
</tbody>
</table>

### Table 9 Scale–rotation–translation synthesis results for problem 4 (ArcofEllipse)

<table>
<thead>
<tr>
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<th></th>
<th></th>
<th></th>
<th></th>
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<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.387188</td>
<td>-0.558232</td>
<td>-0.558645</td>
<td>-1.420063</td>
<td>0.0319</td>
<td>0.0671</td>
<td>0.1746</td>
<td>0.3648</td>
<td>0.1968</td>
</tr>
</tbody>
</table>
for problem 4 in Ref. [30]. The synthesis mechanism must be mirrored with respect to \( Y \)-axis to obtain the final synthesis mechanism, as shown in Fig 11.

### 6.5 Problem 5: GLetter Curve With 40 Unequally Spaced Discrete Points.

The target points for the GLetter curve are shown in Table 10. The number of evaluations of the objective function, the synthesis design variables, the Euclidean distances of the FDs for the three shape signature, and the value of the objective function are shown in Table 11, together with the synthesis solutions obtained by Lin [30] and Buskiewicz [17]. The scale–rotation–translation synthesis results are shown in Table 12. Figure 12 shows the desired path and the coupler curve obtained.

**Table 10** Target points for problem 5 (GLetter)

<table>
<thead>
<tr>
<th>Point</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Y )</td>
<td>12.25</td>
<td>12.094</td>
<td>11.703</td>
<td>11.046</td>
<td>10.099</td>
<td>8.868</td>
<td>7.419</td>
<td>5.917</td>
</tr>
</tbody>
</table>

**Table 11** Shape synthesis results for problem 5 (GLetter)

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( N^o )</td>
<td>100,000</td>
<td>1,600,000</td>
<td>334,070</td>
<td>828,795</td>
</tr>
<tr>
<td>( r_1 )</td>
<td>7.033680</td>
<td>19.20775</td>
<td>15.521</td>
<td>17.3559</td>
</tr>
<tr>
<td>( r_2 )</td>
<td>1</td>
<td>1.006594</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( r_3 )</td>
<td>5.478555</td>
<td>6.479871</td>
<td>12.76</td>
<td>13.005</td>
</tr>
<tr>
<td>( r_4 )</td>
<td>5.188789</td>
<td>15.5133</td>
<td>13.0685</td>
<td>5.3043</td>
</tr>
<tr>
<td>( r_{xy} )</td>
<td>2.231443</td>
<td>1.726245</td>
<td>1.5126</td>
<td>1.0003</td>
</tr>
<tr>
<td>( h_20 )</td>
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<td>2.786092</td>
<td>6.6182</td>
<td>9.5526</td>
</tr>
<tr>
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<td>-10.48661</td>
<td>11.493</td>
<td>17.498</td>
</tr>
<tr>
<td>( \theta_0 )</td>
<td>5.364521</td>
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<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( E_{cc} )</td>
<td>0.0401</td>
<td>0.2373</td>
<td>0.8884</td>
<td>0.2585</td>
</tr>
<tr>
<td>( E_{cd} )</td>
<td>0.0149</td>
<td>0.0062</td>
<td>0.5067</td>
<td>0.0830</td>
</tr>
<tr>
<td>( E_{tc} )</td>
<td>0.0747</td>
<td>0.1295</td>
<td>0.5302</td>
<td>0.4180</td>
</tr>
<tr>
<td>( f_{obj} )</td>
<td>0.0432</td>
<td>0.0531</td>
<td>0.6418</td>
<td>0.2532</td>
</tr>
</tbody>
</table>

**Table 12** Scale–rotation–translation synthesis results for problem 5 (GLetter)

<table>
<thead>
<tr>
<th></th>
<th></th>
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<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( r_2 )</td>
<td>1.030464</td>
<td>2.165309</td>
<td>3.187583</td>
<td>3.223374</td>
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<tr>
<td>( x_0 )</td>
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<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>( y_0 )</td>
<td>-2</td>
<td>-2</td>
<td>-2</td>
<td>-2</td>
</tr>
<tr>
<td>( f_{obj} )</td>
<td>0.0432</td>
<td>0.0531</td>
<td>0.6418</td>
<td>0.2532</td>
</tr>
</tbody>
</table>
comprises the CC, CD, and TCA signatures is proposed because it
value of the magnitude of the FD. The combined discrete FD that
be achieved by dividing the magnitude of the FD by the maximum
formation and using the magnitude of the FD. Scale invariance can
ence the FD. One can achieve rotation invariance and make the
offers better immunity to the problems of the invariance of the
problems may be a subject of future work.

Acknowledgment
The author would like to express their appreciation for the
insightful and constructive comments of the associate editor and
referees which have helped to improve the quality of this manu-
script. This study was supported by the National Science Council
(NSC) of the Republic of China (Taiwan) under the contract NSC
102-2221-E-237-001.

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Shape-Changing Rigid-Body Mechanisms for Morphing Aircraft Wings,”
p. 011007.
sis for Finnitely Separated Positions Using Geometric Constraint Programming,”
Approach to Solving the Problem of Optimum Synthesis of a Six-Bar Double
“Kinematic Synthesis for Infinitesimally and Multiply Separated Positions
Manipulators With Arbitrary Topology Using Shape Representation and Simu-

7 Conclusions
The discrete FD is a very good shape descriptor, because it
offers better immunity to the problems of the invariance of the
similarity transform, the starting point, and normalization. Except
for the DC component, the translation of the shape does not influ-
ence the FD. One can achieve rotation invariance and make the
FD independent of the starting point by ignoring the phase infor-
mation and using the magnitude of the FD. Scale invariance can
be achieved by dividing the magnitude of the FD by the maximum
value of the magnitude of the FD. The combined discrete FD that
comprises the CC, CD, and TCA signatures is proposed because it
can capture greater similarity of shape. The mean value of the
three Euclidean distances corresponding to the three feature vec-
tors between the desired and generated curves is used as the simi-
arity metric and the objective function for shape optimization.
With the help of the determination of analogous points, the
proposed shape synthesis method using the combined FD can be
applied to synthesis problems with either equally spaced or un-
equally spaced target points. The proposed two-phase synthesis
method, which is based on a combined discrete FD, successfully
solves the challenging path generation problems with a relatively
small number of evaluations of the objective function. The mean
value of the three Euclidean distances corresponding to the three
feature vectors obtained using the proposed combined FD shows a
decline of about 39.8%, 55.8%, 26.0%, 76.3%, and 18.6% com-
pared with that obtained using the one-phase synthesis method for
the triangle, asteroid, ParaCurve, ArcoEllipse, and Glitter shapes,
respectively. In general, a more accurate path shape can be
obtained using the combined FD than with the one-phase synthe-
sis method. The obtained coupler curves approximate the desired
paths quite well. With the proposed two-phase synthesis method,
one can avoid using input angles as design variables as well as the
constraint of the rotation sequence of the input angle, which may
influence search quality (solution accuracy), sometimes resulting
in an order defect problem. Moreover, the proposed method can
recognize the mirror reflection of a desired curve.

The use of wavelet descriptors for the challenging path synthe-
sis problems may be a subject of future work.

Fig. 12 Desired path (GLetter) and the coupler curve obtained
using the proposed two-phase synthesis method for problem 5

Fig. 13 Desired path (GLetter) and the coupler curve obtained
by Lin [30] using the one-phase synthesis method for
problem 5

using the proposed two-phase synthesis method. For the purpose
of comparisons, the desired path and the coupler curve obtained
by Lin [30] using the one-phase synthesis method are shown in
Fig. 13: the scale is the same. The present coupler curve approxi-
mates the desired path quite well. The shape similarity of the syn-
thesis results obtained using the proposed two-phase synthesis is
not obviously superior to that obtained using the one-phase
synthesis method for problem 5 in Ref. [30].